Rank of a tensor :→ The rank of a tensor is equal to the number of suffixes or the indices attached to it. The rank of a tensor when raised as power the number of the dimensions gives the number of components of the tensor. Thus a tensor of rank N is three dimensional space has 3^N components and four dimensional space it has 4^n components. Therefore rank of tensor gives the number of the mode of changes of a physical quantity when passing from one system to another one which is in rotation relative to the first. Thus a quantity that does not changed when the axes are rotated is a tensor of zero rank, since the number of mode of the change is zero. These quantities are scalars. The quantities which transform according to the equation.

$$F_a' = \sum_{b=1}^{n} T_{ab} F_b \quad \text{--------} \quad (1)$$

here only one index and hence there is only one way of transformation, so these quantities may be taken as the tensor of rank one. But according to the definition of the transformation (1) these quantities are known as vectors. Hence the vectors are treated as the tensors of rank one.

—x—

Analytic function → A function is said to be analytic at a point $z = a$ if it is differentiable at 'a' and is also differentiable at every point in the neighbourhood of 'a'.

The condition for analyticity is therefore very severe. $f(z) = z^2$, e^z, $\sin z$ are some examples of analytic functions.

The function $f(z) = \frac{1}{z}$ is analytic everywhere except at $z=0$. On the other hand the function $f(z) = |z|^2$ is not analytic at any point - not even at the point $a=0$, although it is differentiable. If a function is analytic everywhere in the entire z-plane, it is called an entire function. Analytic function is sometimes referred to as a holomorphic function ~~of~~, singular function.

A function may be differentiable in a domain D except for a finite number of points. These points are called the singularities or singular points of the function in the domain D.

→ Cauchy - Riemann Differential Equation :→

Necessary Conditions for $w = f(z)$ to be analytic :

The necessary condition for $w=f(z) = u(x,y) + iv(x,y)$ to be analytic ie. differentiable at any point $z = x+iy$ of its domain D is that the four partial derivatives u_x, u_y and v_y should exists and satisfy the Cachy - Riemann partial differential equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$\text{or,} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof → Let $f(z) = u(x,y) + iv(x,y)$ be analytic at any point z of its domain.

$\therefore f'(z) = \underset{\partial z \to z}{Lt} \dfrac{f(z+\partial z) - f(z)}{\partial z}$ exists and is unique

ie. it is independent of the path along which $\partial z \to 0$.

$\because z = x + iy$

$\therefore \partial z = \partial x + i\partial y$

As $\partial z \to 0$, then $\partial x \to 0$ and $\partial y \to 0$ accordingly.

$$\therefore f'(z) = \lim_{\substack{\partial x \to 0 \\ \partial y \to 0}} \frac{[u(x+\partial x, y+\partial y) + iv(x+\partial y, y+\partial y)] - [u(x,y) + iv(x,y)]}{\partial x + \partial y}$$

$$\text{or, } f'(z) = \lim_{\substack{\partial x \to 0 \\ \partial y \to 0}} \left[\frac{u(x+\partial x, y+\partial y) - u(x,y)}{\partial x + i\partial y} + i\, \frac{v(x+\partial x, y+\partial y) - (x,y)}{\partial x + i\partial y} \right] \quad (1)$$

Now, let us consider two possible approches in which $\partial z \to 0$. In the first case let us take ∂z to be purely real so that $\partial z = \partial y = 0$ and $\partial x \to 0$

$\therefore$ From (1), we get :

$$f'(z) = \lim_{\partial x \to 0} \left[\frac{u(x+\partial x, y) - u(x,y)}{\partial x} + i\, \frac{v(x+\partial x, y) - v(x,y)}{\partial x} \right]$$

$$\text{or, } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \quad (2)$$

Since $f'(z)$ exists therefore the above limits exists which in other words means that u_x & v_x exist. In the second case let $\partial z \to 0$ along the imaginary axis so that ∂z is purely imaginary and hence $\partial z = i\partial y$, and $\partial x = 0$ and $\partial y \to 0$.

$\therefore$ From (1), we have :

$$f'(z) = \lim_{\partial y \to 0} \left[\frac{u(x, y+\partial y) - u(x,y)}{i\partial y} + i\, \frac{v(x, y+\partial y) - v(x,y)}{i\partial y} \right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -iu_y + v_y \quad (3)$$

Since $f'(z)$ exists, therefore, the above limit exists which in another words means that u_x and v_y should also exist.

Also by defn, we know that the limit should be unique and hence the two limits obtained in (2) and (3) should be identical.

$$\therefore \quad u_x + i v_x = -i u_y + v_y$$

Equating real and imaginary parts, we get

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \quad \text{---} \quad (4)$$

$$\text{or,} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} \quad (5)$$

Equations (4) and (5) are called Cauchy - Riemann equations. Equations (4) and (5) are necessary conditions for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic, these equations are called the cauchy - Riemann differential eqns. Thus we find that the necessary conditions for the function $f(z)$ to be analytic at the point z is that all the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$ should exist and satisfy the cauchy - Riemann differential equations at the point $z=(x,y)$.

→ Potential at a point outside a charged sphere using Laplace's equation:

Let us consider a charged metal sphere of radius a and at a potential V_a and solve the Laplace's equation for the region outside the sphere. Due to the spherical symmetry the potential V is a function of 'r' only and the Laplace's equation takes the form

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=0 \quad ----- (1)$$

This equation has to be solved in such a manner that $V=0$ at infinity at $V=V_a$ at the surface of the sphere.

$\therefore$ The above equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=0$$

or, $$\int d\left(r^2\frac{\partial v}{\partial r}\right)=\int 0$$

$\therefore$ $$r^2\frac{dv}{dr}=c_1$$, a constt. of integration.

or, $$\int dv = c_1\int\frac{dr}{r^2}$$

$$\therefore V=-\frac{c_1}{r}+c_2 \quad ----- (2)$$

where c_2 is another constt. of integration.

This is the general expression for the potential.

(i) At $r=\infty$, $V=0$

$\therefore$ equation (2) gives, $c_2=0$

$$\therefore V=\frac{-c_1}{r} \quad ----- (3)$$

(ii) At $r=a$, $V=V_a$

$\therefore$ Equation (3) gives: $V_a=-\frac{c_1}{a}$

$$\therefore c_1=-aV_a$$

Putting the values of c_1 and c_2 in eqn (2), we get:

$$V=\left(\frac{a}{r}\right)V_a \quad ----- (4)$$

This is the potential in the region surrounding the charged sphere. If a charge Q is given to a metal sphere of radius 'a', then

$$V_a=\frac{1}{4\pi\epsilon}\cdot\frac{Q}{a} \quad \therefore \boxed{V=\frac{a}{r}\cdot\frac{1}{4\pi\epsilon}\cdot\frac{Q}{a}=\frac{1}{4\pi\epsilon}\cdot\frac{Q}{r}} \quad (5)$$

which is well known relation.